Frobenius number of a linear Diophantine equation

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FROBENIUS NUMBER OF A LINEAR DIOPHANTINE EQUATION

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ABSTRACT. We denote by $\mathbb{N}_0$ the set of nonnegative integers. Let $d \geq 1$ and $A = \{a_1, \ldots, a_d\}$ a set of positive integers. For every $n \in \mathbb{N}_0$, we write $s(n)$ for the number of solutions $(x_1, \ldots, x_d) \in \mathbb{N}_0^d$ of the equation $a_1x_1 + \cdots + a_dx_d = n$. We set $g(A) = \sup\{n \mid s(n) = 0\} \cup \{-1\}$ the Frobenius number of A. Let $S(A)$ be the subsemigroup of $(\mathbb{N}_0, +)$ generated by A. We set $S'(A) = \mathbb{N}_0 \backslash S(A)$, $N'(A) = \mathrm{Card} S'(A)$ and $N(A) =$ Card $S(A) \cap \{0, 1, .., g(A)\}$. Let p be a multiple of lcm(A) and $F_p(t) = \prod_{i=1}^d \sum_{j=0}^{\frac{p}{a_i}-1} t^{ja_i}$. We give an upper bound for $g(A)$ and reduction formulas for $g(A), N'(A)$ and $N(A)$. Characterizations of these invariants as well as numerical symmetric and pseudo-symmetric semigroups in terms of $F_p(t)$, are also obtained.

1 INTRODUCTION

We denote by $\mathbb{N}_0$ (resp. $\mathbb{N}$) the set of nonnegative (resp. positive) integers. Let $d \in \mathbb{N}$ and $A = \{a_1, \ldots, a_d\} \subset \mathbb{N}$. We set $\rho = \gcd(A)$ and $l = \mathrm{lcm}(A)$. For every $n \in \mathbb{N}_0$, we write $s(n)$ for the number of solutions $(x_1, \ldots, x_d) \in \mathbb{N}_0^d$ of the equation $a_1x_1 + \cdots + a_dx_d = n$. We set $g(A) = \sup\{n \mid s(n) = 0\} \cup \{-1\}$ the Frobenius number of A. Let $S(A)$ be the subsemigroup of $(\mathbb{N}_0, +)$ generated by A, $S'(A) = \mathbb{N}_0 \setminus S(A)$, $N'(A) =$ Card S'(A) and $N(A) =$ Card S(A) $\cap \{0, 1, .., \mathrm{g(A)}\}$. We say that $S(A)$ is symmetric (resp. pseudo-symmetric) if $\gcd(A) = 1$ and $N'(A) = N(A)$ (resp. $N'(A) = N(A) + 1$). The generating function of the $s(n)$ is

$$\Phi(t) = \frac{1}{\prod_{i=1}^d (1 - t^{a_i})}.$$

Indeed, we have

$$\frac{1}{\prod_{i=1}^d (1 - t^{a_i})} = \prod_{i=1}^d \sum_{j \geq 0} t^{ja_i} = \sum_{n \in S(A)} s(n)t^n.$$

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For $p \in l\mathbb{N}$, we define the Frobenius polynomial

$$F_p(t) = \prod_{i=1}^{d} \sum_{j=0}^{\frac{p}{a_i}-1} t^{ja_i} = \frac{(1-t^p)^d}{\prod_{i=1}^{d}(1-t^{a_i})}$$

and we write

$$\Phi(t) = \frac{F_p(t)}{(1-t^p)^d}. \tag{1}$$

In theorem 3.1 we give formulas for $g(A), N'(A)$ and $N(A)$ in terms of $F_p(t)$. As a consequence we obtain an upper bound for the Frobenius number (corollary 3.2) which improves the upper bound given by Chrzastowski-Wachtel and mentioned in [9]. A characterization of numerical symmetric and pseudo-symmetric semigroups (corollary 3.4) is also obtained. In theorem 3.7 we prove reduction formulas for $g(A)$, $N'(A)$ and $N(A)$. The first one generalizes a Raczunas and Chrzastowski-Wachtel theorem [9]. As a consequence (corollary 3.10) we obtain a generalization of a Rödseth formula [10]. It is known that the Hilbert function of a graded module over a polynomial graded ring as well as $s(n)$ are numerical quasi-polynomial functions. In examples 4.9 and 4.10 we give a description of these functions in terms of the Frobenius polynomial.

2 PRELIMINARIES

Given $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t, t^{-1}]$ and an integer $p \geq 1$, there exists a unique sequence $Q_0, \ldots, Q_{p-1} \in \mathbb{Q}[t, t^{-1}]$ such that $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p)$. Namely, $Q_r(t) = \sum_k q_{r+pk} t^k$. The Q_r are called the *p-components* of Q. We denote by $\omega(Q) = \inf\{j \mid q_j \neq 0\}$ the valuation of Q and $\deg(Q) = \sup\{j \mid q_j \neq 0\}$ the degree of Q, with $\omega(0) = +\infty$ and $\deg(0) = -\infty$. The following invariants will be associated with Q

$$\begin{aligned}
\omega_p(Q) &= \sup\{\omega(t^r Q_r(t^p)) \mid 0 \leq r \leq p-1\} \text{ the } p\text{-valuation of } Q. \\
\delta_p(Q) &= \inf\{\deg(t^r Q_r(t^p)) \mid 0 \leq r \leq p-1\} \text{ the } p\text{-degree of } Q. \\
\Omega_p(Q) &= \sum_{r=0}^{p-1} \omega(Q_r). \\
\Delta_p(Q) &= \sum_{r=0}^{p-1} \deg(Q_r).
\end{aligned}$$

Thus we have
$\omega_p(Q) = +\infty = \Omega_p(Q)$ and $\delta_p(Q) = -\infty = \Delta_p(Q)$ if $Q_r = 0$ for some r.

We fix an integer $n \in \mathbb{Z}$ and we set

$$\widehat{Q}(t) = t^n Q(t^{-1}).$$

So we have $\widehat{\widehat{Q}} = Q$ and

$$\deg(Q) + \omega(\widehat{Q}) = n = \deg(\widehat{Q}) + \omega(Q) \text{ if } Q \neq 0. \tag{2}$$

The p-components $\widehat{Q}_r$ of $\widehat{Q}$ can be deduced from the p-components of Q. Namely, we write $n = p\lambda + \gamma$ with $0 \le \gamma < p$, so we get

$$\widehat{Q}(t) = \sum_{r=0}^{p-1} t^{p\lambda+\gamma-r} Q_r(t^{-p}) = \sum_{r=0}^{\gamma} t^{\gamma-r}(t^p)^\lambda Q_r(t^{-p}) + \sum_{r=\gamma+1}^{p-1} t^{p+\gamma-r}(t^p)^{\lambda-1} Q_r(t^{-p}).$$

It follows from the uniqueness of the p-components that

$$\widehat{Q}_r(t) = t^\lambda Q_{\gamma-r}(t^{-1}) \text{ for } 0 \le r \le \gamma \tag{3}$$

and

$$\widehat{Q}_r(t) = t^{\lambda-1} Q_{p+\gamma-r}(t^{-1}) \text{ for } r > \gamma. \tag{4}$$

So we obtain

$$\widehat{Q}_r = 0 \Leftrightarrow Q_{\gamma-r} = 0 \text{ for } 0 \le r \le \gamma \tag{5}$$

and

$$\widehat{Q}_r = 0 \Leftrightarrow Q_{p+\gamma-r} = 0 \text{ for } r > \gamma. \tag{6}$$

If $\widehat{Q}_r \neq 0$, we also deduce from (2)-(4) that

$$\lambda = \deg(\widehat{Q}_r) + \omega(Q_{\gamma-r}) \text{ when } 0 \le r \le \gamma \tag{7}$$

and

$$\lambda - 1 = \deg(\widehat{Q}_r) + \omega(Q_{p+\gamma-r}) \text{ when } r > \gamma. \tag{8}$$

Moreover, writing $n = p\lambda + r + (\gamma - r) = p(\lambda-1) + r + (p+\gamma-r)$ we get

$$n = \deg(t^r \widehat{Q}_r(t^p)) + \omega(t^{\gamma-r} Q_{\gamma-r}(t^p)) \text{ for } 0 \le r \le \gamma$$

and

$$n = \deg(t^r \widehat{Q}_r(t^p)) + \omega(t^{p+\gamma-r} Q_{p+\gamma-r}(t^p)) \text{ for } r > \gamma.$$

Hence

$$n = \delta_p(\widehat{Q}) + \omega_p(Q) = \delta_p(Q) + \omega_p(\widehat{Q}). \tag{9}$$

Furthermore, using (7) and (8) we get

$$\sum_{r=0}^{\gamma} \Big(\deg(\widehat{Q}_r) + \omega(Q_{\gamma-r})\Big) + \sum_{r=\gamma+1}^{p-1} \Big(\deg(\widehat{Q}_r) + \omega(Q_{p+\gamma-r})\Big)$$

$$= (\gamma+1)\lambda + (p-\gamma-1)(\lambda-1) = n - p + 1.$$

It follows that

$$\Delta_p(\widehat{Q}) + \Omega_p(Q) = n - p + 1 = \Delta_p(Q) + \Omega_p(\widehat{Q}). \tag{10}$$

Given $m,\, j \in \mathbb{Z}$, we consider the following polynomials

$$N_{m,j}(t) = \frac{1}{(m-1)!} \prod_{i=1}^{m-1} (t - j + i) \text{ if } m > 1, N_{m,j}(t) = 0 \text{ if } m \le 0 \text{ and } N_{1,j}(t) = 1.$$

For $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t, t^{-1}]$ such that $Q(1) \neq 0$, we define

$$V_m(Q, t) = \sum_j q_j N_{m,j}(t).$$

Furthermore, let $Q_0, \ldots, Q_{p-1} \in \mathbb{Q}[t, t^{-1}]$ be the p-components of Q. We consider the polynomials $U_0, \ldots, U_{p-1} \in \mathbb{Q}[t, t^{-1}]$ defined as follows $U_r = 0$ if $Q_r = 0$ and $Q_r(t) = (1-t)^{i_r} U_r(t)$ with $U_r(1) \neq 0$ otherwise. For all $0 \leq r \leq p-1$, we put $m_r = m - i_r$ and we define the function

$$H_m(Q, .) : \mathbb{Z} \to \mathbb{Q} \text{ by } H_m(Q, r + pk) = V_{m_r}(U_r, k).$$

In order to illustrate these definitions we give the following examples.

EXAMPLE 2.1 Let $Q(t) = F_{12} = \frac{(1-t^{12})^2}{(1-t^2)(1-t^3)} = 1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8 + 2t^9 + 2t^{10} + 2t^{11} + t^{12} + 2t^{13} + t^{14} + t^{15} + t^{16} + t^{17} + t^{19}$.
We take $p = 12, n = 19$ and $m = 2$.
We write $Q(t) = (1 + t^{12}) + t(2t^{12}) + t^2(1 + t^{12}) + t^3(1 + t^{12}) + t^4(1 + t^{12}) + t^5(1 + t^{12}) + 2t^6 + t^7(1 + t^{12}) + 2t^8 + 2t^9 + 2t^{10} + 2t^{11}$.
We see that the 12-components of $Q(t)$ are $Q_0(t) = Q_2(t) = Q_3(t) = Q_4(t) = Q_5(t) = Q_7(t) = (1 + t), Q_1(t) = 2t$ and $Q_6(t) = Q_8(t) = Q_9(t) = Q_{10}(t) = Q_{11}(t) = 2$.
We also have
$\widehat{Q}(t) = t^{19} Q(t^{-1}) = Q(t)$.
$\omega_{12}(Q) = 13,\ \delta_{12}(Q) = 6,\ \Omega_{12}(Q) = 1,\ \Delta_{12}(Q) = 7$.
$N_{2,0}(t) = t + 1, N_{2,1}(t) = t$.
$U_r = Q_r$ for all r.
$V_2(U_r, t) = 2t + 1$ for $r \in \{0, 2, 3, 4, 5, 7\}$, $V_2(U_1, t) = 2t$ and $V_2(U_r, t) = 2(t + 1)$ for $r \in \{6, 8, 9, 10, 11\}$.
We obtain $H_2(Q, 12k + r) = 2k + 1$ for $r \in \{0, 2, 3, 4, 5, 7\}$, $H_2(Q, 12k + 1) = 2k$ and $H_2(Q, 12k + r) = 2(k + 1)$ for $r \in \{6, 8, 9, 10, 11\}$.

EXAMPLE 2.2 Let $Q(t) = F_6(t) = 1 + t^2 + t^3 + t^4 + t^5 + t^7 = \frac{(1-t^6)^2}{(1-t^2)(1-t^3)}$.
We take $p = 6, n = 7$ and $m = 2$.
We obtain
$\omega_6(Q) = 7,\ \delta_6(Q) = 0,\ \Omega_6(Q) = 1,\ \Delta_6(Q) = 1$.
$U_r = Q_r$ for all r.
$N_{2,0}(t) = t + 1, N_{2,1}(t) = t$.
$V_2(U_r, t) = t + 1$ for $r \in \{0, 2, 3, 4, 5\}$ and $V_2(U_1, t) = t$.
$H_2(Q, 6k + r) = k + 1$ for $r \in \{0, 2, 3, 4, 5\}$ and $H_2(Q, 6k + 1) = k$.
We observe that $H_2(F_6, .) = H_2(F_{12}, .)$.

Given $\Phi(t) \in \mathbb{Q}[[t, t^{-1}]]$, we write $\Phi(t) = \sum_n \varphi(n) t^n$ and we introduce

$$\begin{aligned}
g(\Phi) &= \sup\{n \mid \phi(n) = 0\}. \\
S'(\Phi) &= \{n \geq 0 \mid \varphi(n) = 0\}. \\
S(\Phi) &= \{0 \leq n \leq g(\Phi) \mid \varphi(n) \neq 0\}. \\
N'(\Phi) &= \text{Card } S'(\Phi). \\
N(\Phi) &= \text{Card } S(\Phi).
\end{aligned}$$

LEMMA 2.3 *Given* $m \in \mathbb{Z}$ *and* $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t, t^{-1}]$ *such that* $Q(1) \neq 0$, *we consider* $\Phi(t) = \sum_n \varphi(n) t^n$ *the expansion of* $(1-t)^{-m} Q(t)$ *as a formal power series. Then, the following conditions hold*

1. $\varphi(n) = V_m(Q, n)$ *for all* $n > \deg(Q) - m$.
2. *We suppose that* $m > 0$ *and* $Q(t)$ *has nonnegative coefficients. Then,*
   (a) $\varphi(n) = 0 \Leftrightarrow n < \omega(Q)$.
   (b) $g(\Phi) = \omega(Q) - 1$.
   (c) $N'(\Phi) = \max\{\omega(Q), 0\}$. In particular, $N'(\Phi) = \omega(Q)$ if $Q(t) \in \mathbb{Q}[t]$.

PROOF. 1. Suppose $m > 0$. We have $\Phi(t) = (1-t)^{-m} Q(t) = (\sum_j q_j t^j) \sum_{j \geq 0} \binom{j+m-1}{m-1} t^j$. So $\varphi(n) = \sum_{j=\omega(Q)}^{n} q_j \binom{n-j+m-1}{m-1}$. Moreover, we have

$$\binom{n-j+m-1}{m-1} = \frac{1}{(m-1)!} \prod_{i=1}^{m-1} (n-j+i) \text{ if } n \geq j.$$

Hence $\varphi(n) = V_m(Q, n)$ if $n \geq \deg(Q)$, in particular, the statement is true for $m = 1$. Now, suppose $m > 1$ and $\deg(Q) - m < n < \deg(Q)$ then $-m < n - \deg(Q) \leq n - j < 0$ for all j such that $n < j \leq \deg(Q)$. It follows that there exists $1 \leq i \leq m-1$ such that $n - j + i = 0$ thus $N_{m,j}(n) = 0$. So we can write

$$V_m(Q, n) = \sum_{j=\omega(Q)}^{n} q_j N_{m,j}(n) = \sum_{j=\omega(Q)}^{n} q_j \binom{n-j+m-1}{m-1} = \varphi(n).$$

Furthermore, if $m \leq 0$ then $\varphi(n) = 0$ for $n > \deg(Q) - m$ because $\Phi(t) \in \mathbb{Q}[t, t^{-1}]$ and $\deg(Q) - m = \deg \Phi(t)$.
2. Follows from the fact that $\varphi(n) = \sum_{j=\omega(Q)}^{n} q_j \binom{n-j+m-1}{m-1} > 0$ if $n \geq \omega(Q)$ and $\varphi(n) = 0$ if $n < \omega(Q)$ $\square$

THEOREM 2.4 *Let* $m \in \mathbb{Z}$ *and* $p \in \mathbb{N}$. Given $Q(t) = \sum_j q_j t^j \in \mathbb{Q}[t, t^{-1}]$ *such that* $Q(1) \neq 0$, *we consider* $\Phi(t) = \sum_n \varphi(n) t^n$ *the expansion of* $(1-t^p)^{-m} Q(t)$ *as a formal power series. Then the following conditions hold*

1. $\varphi(n) = H_m(Q, n)$ *for all* $n > \deg(Q) - mp$.
2. *We suppose that* $m > 0$ *and* $Q(t)$ *has nonnegative coefficients. Then,*
   (a) $\varphi(pk + r) = 0 \Leftrightarrow k < \omega(Q_r)$.
   (b) $g(\Phi) = \omega_p(Q) - p = \deg(Q) - p - \delta_p(\hat{Q})$ *where* $\hat{Q}(t) = t^{\deg(Q)} Q(t^{-1})$.
   (c) $N'(\Phi) = \sum_{r=0}^{p-1} \max\{\omega(Q_r), 0\}$.
   In particular, $N'(\Phi) = \Omega_p(Q)$ if $Q(t) \in \mathbb{Q}[t]$.

PROOF. We write $\Phi(t) = \sum_{r=0}^{p-1} t^r (1-t^p)^{-m} Q_r(t^p) = \sum_{r=0}^{p-1} t^r \, (1-t^p)^{-m_r} U_r(t^p) = \sum_{r=0}^{p-1} t^r \Phi_r(t^p)$ where $\Phi_r(t) = (1-t^p)^{-m_r} U_r(t^p) = \sum_k \varphi_r(k) t^k$. It follows from lemma 2.3.1, that $\varphi(pk+r) = \varphi_r(k) = V_{m_r}(U_r, k)$ for all $k > \deg(U_r) - m_r$. Therefore, $\varphi(n) = H_m(Q, n)$ for $n > \deg(Q) - pm$ because $n = pk + r > \deg(Q) - pm \geq$

$p(\deg(Q_r)-m)+r \Rightarrow k > \deg(Q_r)-m = \deg(U_r)-m_r$.
2 (a) follows from lemma 2.3.2 (a).
b) We have $g(\Phi) = \max\{pg(\Phi_r)+r \mid 0 \le r \le p-1\} = \max\{p(\omega(Q_r)-1)+r \mid 0 \le r \le p-1\} = \omega_p(Q)-p$. Moreover, if $Q_r \neq 0$ for all r we have $\omega_p(Q)-p = \deg(Q)-p-\delta_p(\hat{Q})$ by (9). Since $\omega_p(Q) = +\infty = -\delta_p(\hat{Q})$ if $Q_r = 0$ for some r, the equality is still true in this case.
c) Follows from lemma 2.3.2 (c) □

LEMMA 2.5 *Let $\xi = e^{\frac{2i\pi}{p}}$ be a primitive p-th root of unity and $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p) \in \mathbb{Q}[t,t^{-1}]$. Then, the following conditions are equivalent*

1. $Q(\xi^j) = 0$ *for* $0 < j < p$.
2. $Q(1) = pQ_r(1)$ *for* $0 \le r \le p-1$.

PROOF. By successive substitutions of $1, \xi, \ldots, \xi^{p-1}$ for t in $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p)$ we obtain a Vandermonde linear system $\sum_{r=0}^{p-1} \xi^{rj} Q_r(1) = Q(\xi^j)$ for $j = 0, \ldots, p-1$. If $Q(\xi) = \cdots = Q(\xi^{p-1}) = 0$, the unique solution is $Q_r(1) = \frac{1}{p}Q(1)$ for every $0 \le r \le p-1$. Conversely, if $\frac{Q(1)}{p}$ is the common value of the $Q_r(1)$ then $\frac{Q(1)}{p}\sum_{r=0}^{p-1} \xi^{rj} = 0 = Q(\xi^j)$ for $j = 1, \ldots, p-1$ □

LEMMA 2.6 *Let p, q, u be positive integers and $Q(t), K(t) \in \mathbb{Q}[t,t^{-1}]$ such that $p = qu$ and $K(t^u) = Q(t)$. We denote by Q_r (resp. K_s) the p-components of Q (resp. the q-components of K). Then,*

1. $Q_{su} = K_s$ *and* $Q_r = 0$ *for all* $r \notin u\mathbb{Z}$.
2. *We set* $\xi = e^{\frac{2i\pi}{p}}$, *then the following conditions are equivalent*
   (a) $Q(\xi^j) = 0$ *for* $0 < j < q$.
   (b) $Q(\xi^q) = qQ_r(1) = K(1)$ *for all* $r \in u\mathbb{Z}$.

PROOF. We can write $Q(t) = K(t^u) = \sum_{s=0}^{q-1} t^{us} K_s(t^p)$. It follows from the uniqueness of the Q_r that $Q_{su} = K_s$ for $0 \le s < q$. Now, $Q(\xi^q) = K(1)$ and $Q(\xi^j) = K(\alpha^j)$ with $\alpha = e^{\frac{2i\pi}{q}} = \xi^u$. We apply lemma 2.5 □

For every $p \in l\mathbb{N}$, we set $F_p(t) = \prod_{i=1}^d \sum_{j=0}^{\frac{p}{a_i}-1} t^{ja_i}$ the *Frobenius polynomial* of A. We write $F_{p,r}$ for the p-components of F_p. It is easy to see that for $n = \deg(F_p) = pd - \sum_{i=1}^d a_i$, we have $\widehat{F}_p(t) = t^n F_p(t^{-1}) = F_p(t)$. Let us write $p = q\rho$ and $a_i = b_i\rho$ for all $1 \le i \le d$, where $\rho = \gcd(A)$. So we can write $F_p(t) = K(t^\rho)$ with

$$K(t) = \frac{(1-t^q)^d}{\prod_{i=1}^d (1-t^{b_i})}.$$

Moreover, for $0 < j < q$ the number $\xi^j = e^{\frac{2ij\pi}{q}}$ is a root of $\prod_{i=1}^d (1-t^{b_i})$ of multiplicity $< d$ because $\gcd(b_1, .., b_d) = 1$ whereas ξ^j is a root of $(1-t^q)^d$ of multiplicity $= d$, then $K(\xi^j) = 0$. It follows from lemma 2.6 that $F_{p,r} = K_{\frac{r}{\rho}}$ if $r \in \rho\mathbb{Z}$ and $F_{p,r} = 0$ otherwise. We also deduce that $F_{p,r}(1) = \frac{1}{q}K(1) = \frac{\rho p^{d-1}}{\prod_{i=1}^d a_i}$ if $r \in \rho\mathbb{Z}$ □

3 FROBENIUS NUMBER AND NUMERICAL SEMIGROUPS

In the case of the Frobenius polynomial F_p we set $\omega_p(F_p) = \omega_p(A)$, $\delta_p(F_p) = \delta_p(A)$, $\Omega_p(F_p) = \Omega_p(A)$, $\Delta_p(F) = \Delta_p(A)$.

THEOREM 3.1 *For every $p \in l\mathbb{N}$, we have*

1. $g(A) = \omega_p(A) - p = p(d-1) - \sum_{i=1}^{d} a_i - \delta_p(A) = l(d-1) - \sum_{i=1}^{d} a_i - \delta_l(A)$.
2. $N'(A) = \Omega_p(A) = \Omega_l(A)$.
3. $N(A) = \Delta_p(A) - \delta_p(A) = \Delta_l(A) - \delta_l(A)$.

PROOF. We see that for every $p \in l\mathbb{N}$, the function $\Phi(t) = (1-t^p)^{-d} F_p(t) = \sum_n s(n)t^n$ is the generating function of the $s(n)$ so $g(A) = g(\Phi)$.
1. follows from theorem 2.4.2 (b).
2. follows from theorem 2.4.2 (c).
3. is a consequence of (10)□

COROLLARY 3.2

1. *For every $p \in l\mathbb{N}$, we have*

$$g(A) = p(d-1) - \sum_{i=1}^{d} a_i \text{ if and only if } \delta_p(A) = 0.$$

2. $g(A) = +\infty$ *if and only if* $\rho > 1$.
3. *If* $\rho = 1$, *we have the following upper bound for the Frobenius number*

$$g(A) \leq l(d-1) - \sum_{i=1}^{d} a_i.$$

4. *If there exists h such that $1 \leq h \leq d$ and* $\gcd(a_1, \ldots, a_h) = 1$ *then* $g(A) \leq \mathrm{lcm}(a_1, \ldots, a_h)(h-1) - \sum_{i=1}^{h} a_i$.

REMARK 3.3 The upper bound we give in 3) improves the following inequality

$$g(A) \leq l(d-1)$$

proved by Chrzastowski-Wachtel and mentioned in [9].

COROLLARY 3.4 Suppose $\gcd(A) = 1$. Then the following conditions hold

1. *$S(A)$ is symmetric* $\Leftrightarrow \Delta_p(A) = \Omega_p(A) + \delta_p(A)$ for some $p \in l\mathbb{N} \Leftrightarrow \Delta_p(A) = \Omega_p(A) + \delta_p(A)$ for all $p \in l\mathbb{N}$.
2. *$S(A)$ is peudo-symmetric* $\Leftrightarrow \Delta_p(A) + 1 = \Omega_p(A) + \delta_p(A)$ for some $p \in l\mathbb{N} \Leftrightarrow \Delta_p(A) + 1 = \Omega_p(A) + \delta_p(A)$ for all $p \in l\mathbb{N}$.

We suppose $\gcd(A) = 1$. Let $q_1, .., q_d$ be positive integers such that for all $1 \le i \le d$, q_i is a divisor of $\gcd(a_1, .., a_{i-1}, a_{i+1}, .., a_d)$. So $\gcd(q_i, q_j) = 1$ for $i \ne j$ because $\gcd(A) = 1$. We set $\hat{q} = \prod_{j=1}^d q_j, \hat{q}_i = \prod_{j \ne i} q_j, a_i = b_i \hat{q}_i$ and $B = \{b_1, .., b_d\}$. We have $\gcd(B) = 1$ and $l = \operatorname{lcm}(A) = \hat{q}\operatorname{lcm}(B)$. For $p \in l\mathbb{N}$, we write $p = \hat{q}u$ with $u \in \operatorname{lcm}(B)\mathbb{N}$.

THEOREM 3.5 *The following formulas hold*

1. $\delta_p(A) = \hat{q}\delta_u(B)$.
2. $\omega_p(A) = \hat{q}\omega_u(B) + \sum_{i=1}^d (q_i - 1)a_i$.
3. $\Omega_p(A) = \hat{q}\Omega_u(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.
4. $\Delta_p(A) = \hat{q}\Delta_u(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.

In order to prove this theorem we need a lemma.

LEMMA 3.6 *Let q and c be two positive integers, $B = \{b_1, .., b_{d-1}, c\}$, and $A = \{a_1, .., a_{d-1}, c\}$ where $a_1 = qb_1, .., a_{d-1} = qb_{d-1}$. Suppose* $\gcd(A) = 1$ *and choose* $p \in \operatorname{lcm}(B)\mathbb{N}$ *so* $\gcd(q, c) = 1$ *and* $qp \in \operatorname{lcm}(A)\mathbb{N}$. *Then, the following formulas hold*

1. $\delta_{qp}(A) = q\delta_p(B)$.
2. $\omega_{qp}(A) = q\omega_p(B) + (q-1)c$.
3. $\Omega_{qp}(A) = q\Omega_p(B) + \frac{1}{2}(q-1)(c-1)$.
4. $\Delta_{qp}(A) = q\Delta_p(B) + \frac{1}{2}(q-1)(c-1)$.

PROOF. We denote by

$$F(t) = F_p(t) = \frac{(1-t^p)^d}{(1-t^c)\prod_{i=1}^{d-1}(1-t^{b_i})} = \sum_{r=0}^{p-1} t^r F_r(t^p)$$

the Frobenius polynomial associated with B and

$$G(t) = G_{qp}(t) = \frac{(1-t^{qp})^d}{(1-t^c)\prod_{i=1}^{d-1}(1-t^{a_i})} = \sum_{s=0}^{qp-1} t^s G_s(t^{qp})$$

the Frobenius polynomial associated with A. We see that

$$G(t) = (1 + t^c + .. + t^{(q-1)c})F(t^q) = (1 + t^c + .. + t^{(q-1)c})\sum_{r=0}^{p-1} t^{qr} F_r(t^{qp}).$$

So we obtain

$$G(t) = \sum_{\substack{k=ic+jq \\ 0 \le i \le q-1}} t^k F_j(t^{qp}) = \sum_{\substack{0 \le k=ic+jq \le qp-1 \\ 0 \le i \le q-1}} t^k F_j(t^{qp}) + \sum_{\substack{k > qp-1 \\ 0 \le i \le q-1}} t^{k-qp} t^{qp} F_j(t^{qp})$$

By identification we deduce that $G_s(t^{qp}) = F_j(t^{qp})$ when $s = ic + jq$ and $G_s(t^{qp}) = t^{qp}F_j(t^{qp})$ when $s = ic + jq - qp = ic - (p - j)q$. In particular, we have $\deg(G_s) = \deg(F_j)$ and $\omega(G_s) = \omega(F_j)$ when $s = ic + jq$ and $\deg(G_s) = 1 + \deg(F_j)$ and $\omega(G_s) = 1 + \omega(F_j)$ when $s = ic + jq - qp$. Therefore, for all s which can be written in the form $s = ic + jq$ we get $\deg(t^s G_s(t^{qp})) = ic + jq + qp\deg(F_j)$ and $\omega(t^s G_s(t^{qp})) = ic + jq + qp\omega(F_j)$. For all s which can be written in the form $s = ic + jq - qp$, we get $\deg(t^s G_s(t^{qp})) = ic + jq - qp + qp(1 + \deg(F_j)) = ic + jq + qp\deg(F_j)$ and $\omega(t^s G_s(t^{qp})) = ic + jq - qp + qp(1 + \omega(F_j)$. It follows that $\delta_{qp}(G) = \min\{ic + jq + qp\deg(F_j)\} = q\min\{j + p\deg(F_j)\} = q\delta_p(F)$ and $\omega_{qp}(G) = \max\{ic+jq+qp\omega(F_j)\ \} = (q-1)c+q\max\{j+p\omega(F_j)\ \} = q\omega_p(F)+(q-1)c$. We also have

$$\Omega_{qp}(G) = \sum_{s=ic+jq} \omega(G_s) + \sum_{s=ic+jq-qp} \omega(G_s) = \sum_{s=ic+jq} \omega(F_j) + \sum_{s=ic-jq} (\omega(F_j) + 1)$$

$= q\Omega_p(F) + N'(c, q) = q\Omega_p(F) + \frac{1}{2}(q - 1)(c - 1)$. It follows that
$\Delta_{qp}(G) = \Omega_{qp}(G) + \delta_{qp}(G) = q(\Omega_p(F) + \delta_p(F)) + \frac{1}{2}(q - 1)(c - 1)\square$

PROOF OF THEOREM 3.5. By induction on the number $h = d - k + 1$ such $q_1 = q_2 = .. = q_{k-1} = 1$. If $h = 1$ the result is given by lemma 3.6. Suppose that the result is true when $q_1 = q_2 = .. = q_{k-1} = 1$. We choose $p \in \mathrm{lcm}(A)\mathbb{N}$ and we set $v = \frac{p}{q_k}$, $t_i = q_i$ for $i \neq k$ and $t_k = 1$. Then, we get $\hat{t}_i = \frac{\hat{q}_i}{q_k}$ for all $i \neq k$, $\hat{t}_k = \hat{q}_k$ and $\hat{t} = \frac{\hat{q}}{q_k}$. We also have $\frac{a_i}{q_k} = \frac{b_i\hat{q}_i}{q_k} = b_i\hat{t}_i$ for all $i \neq k$ and $a_k = b_k\hat{t}_k$. We put $c_i = b_i\hat{t}_i$ for all i and $C = \{c_1, .., c_d\}$, thus $a_i = q_k c_i$ for all $i \neq k$ and $a_k = c_k$. It follows from lemma 3.6 and the induction hypothesis that
1) $\delta_p(A) = q_k\delta_v(C) = q_k\hat{t}\delta_u(B) = \hat{q}\delta_u(B)$.
2) $\omega_p(A) = q_k\omega_v(C) + (q_k - 1)c_k = q_k\{\hat{t}\omega_u(B) + \sum_{i=1}^d (t_i - 1)c_i\} + (q_k - 1)c_k = \hat{q}\omega_u(B) + \sum_{i=1}^d (q_i - 1)a_i$.
3)$\Omega_p(A) = q_k\Omega_v(C) + \frac{1}{2}(q_k - 1)(a_k - 1) = q_k\{\hat{t}\Omega_u(B) + \frac{1}{2}(\sum_{i=1}^d (t_i - 1)c_i - \hat{t} + 1)\} + \frac{1}{2}(q_k - 1)(a_k - 1) = \hat{q}\Omega_u(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.
4) $\Delta_p(A) = \Omega_p(A) + \delta_p(A) = \hat{q}\Delta_u(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$ $\square$

THEOREM 3.7 *The following formulas hold*

1. $g(A) = \hat{q}g(B) + \sum_{i=1}^d (q_i - 1)a_i$.
2. $N'(A) = \hat{q}N'(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.
3. $N(A) = \hat{q}N(B) + \frac{1}{2}\left(\sum_{i=1}^d (q_i - 1)a_i - \hat{q} + 1\right)$.

REMARK 3.8 In formula 1) if we take $q_1 = .. = q_{d-1} = 1$ then we obtain a Brauer and Shockley formula [5] and if we take $q_i = \gcd(A\backslash\{a_i\})$ for all i, we obtain a Raczunas and Chrzastowski-Wachtel formula [9]. Moreover formula 2) is a generalization of a Rödseth formula [10] which is obtained for $q_1 = .. = q_{d-1} = 1$.

THEOREM 3.9 *The following conditions hold*

1. *$S(A)$ is symmetric if and only if $S(B)$ is symmetric.*

2. *If* $\hat{q} > 1$ *then* $S(A)$ *is not pseudo-symmetric.*

COROLLARY 3.10 *Suppose there exists* i *such that* $b_i = 1$ (i.e. $a_i = \hat{q}_i$). *Then,* $S(A)$ *is symmetric* and we have

1. (a) $g(A) = \sum_{i=1}^{d}(q_i - 1)a_i - \hat{q}$.

   (b) $N(A) = N'(A) = \frac{1}{2}(\sum_{i=1}^{d}(q_i - 1)a_i - \hat{q} + 1)$.

2. *Suppose, in addition, that* $b_i = 1$ (i.e. $a_i = \hat{q}_i$) *for all* i. *Then, we have*

   (a) $g(A) = l(d-1) - \sum_{i=1}^{d} a_i$.

   (b) $N(A) = N'(A) = \frac{1}{2}(l(d-1) - \sum_{i=1}^{d} a_i + 1)$.

PROOF. Since $1 \in B$, we have $S(B) = \mathbb{N}_0$ then $g(B) = -1$ and $N(B) = N'(B) = 0$. So 1. follows from theorem 3.7. To prove 2., we observe that $q_i a_i = \hat{q} = l = \mathrm{lcm}(A)$ if $a_i = \hat{q}_i$ for all $i\square$

COROLLARY 3.11 *Let* b, d, h, v *be positive integers such that* $b \geq d \geq 2$ *and* $\gcd(b, v) = 1$. *Let* $B = \{b, hb + v, .., hb + (i-1)v, .., hb + (d-1)v\}$, $((b_1, .., b_d)$ *is called an "almost" arithmetic sequence). Then,*
$S(A)$ *is symmetric* $\Leftrightarrow S(B)$ *is symmetric* $\Leftrightarrow d = 2$ or $b \equiv 2 \ \mathrm{mod(d-1)}$.

PROOF. We write $b - 1 = \beta(d-1) + \alpha$ with $0 \leq \alpha < d - 1$, and we use the following known formulas $g(B) = \left(h\left\lfloor\frac{b-2}{d-1}\right\rfloor + h - 1\right)b + bv - v$ [8] and $N'(B) = \frac{1}{2}\{(b-1)(h\beta + v + h - 1) + h\alpha(\beta + 1)\}$ [11] $\square$

EXAMPLE 3.12 Let $A = \{150, 462, 840, 1365\} = \{5(2 \times 3 \times 5), 11(2 \times 3 \times 7), 12(2 \times 5 \times 7), 13(3 \times 5 \times 7)\}$. We set $q_1 = 7, q_2 = 5, q_3 = 3, q_4 = 2$ and $B = \{5, 11, 12, 13\}$. This is an almost arithmetic sequence with $b = 5, v = 1, h = 2, d = 4$. We see that $b \equiv 2 \ \mathrm{mod(d-1)}$ hence $S(B)$ is symmetric and we have $g(B) = 19, N'(B) = N(B) = 10$. Moreover, it follows from theorem 3.9 that $S(A)$ is symmetric. Using theorem 3.7 we obtain $g(A) = 210 \times 19 + 6 \times 150 + 4 \times 462 + 2 \times 840 + 1365 = 9783$. $N'(A) = N(A) = 210 \times 10 + \frac{1}{2}(6 \times 150 + 4 \times 462 + 2 \times 840 + 1365 - 210 + 1) = 4892$.

4 QUASI-POLYNOMIALS

DEFINITION 4.1 *A quasi-polynomial* P *of period* p *and degree* d *is a sequence* $P = (P_0, \ldots, P_{p-1})$ *with* $P_r \in \mathbb{Q}[t]$ *such that* $d = \sup\{\deg(P_r) \mid 0 \leq r \leq p-1\}$.
A quasi-polynomial P is said to be *uniform* if all the P_r have the same degree d and the same leading coefficient $c(P)$. Given a function $h : \mathbb{Z} \to \mathbb{Q}$ and $r \in \mathbb{Z}$, we define $h_r : \mathbb{Z} \to \mathbb{Q}$, $k \mapsto h(pk + r)$. We say that h is a quasi-polynomial function if there exists a quasi-polynomial $P = (P_0, \ldots, P_{p-1})$ such that $h_r(k) = P_r(k)$ for all $k \gg 0$ and $0 \leq r \leq p$. We also say that h is P-quasi-polynomial. It is easily seen that a quasi-polynomial function h has a minimal period and every period of h is a multiple of this minimal period. Furthermore, for a fixed period p, h is a P-quasi-polynomial for a unique sequence $P = (P_0, \ldots, P_{p-1})$. A P-quasi-polynomial h is said to be uniform if P is uniform. We write $\deg(h) = \deg(P)$ and $c(h) = c(P)$. We denote by $F(\mathbb{Z})$ the set of all functions $h : \mathbb{Z} \to \mathbb{Q}$. For every integer $i \geq 0$ we consider the operators E^i and Δ_i, which act as follows:

$(E^i h)(n) = h(n+i)$, $(\Delta_i h)(n) = h(n+i) - h(n)$. We set $E^0 = I$, $E^1 = E$ and $\Delta_1 = \Delta$ so $\Delta = E - I$, $\Delta_0 = 0$ and $\Delta_i = E^i - I$. For $a \geq 0$ and $n \geq 1$, we have $(I + E^a + \cdots + E^{(n-1)a}) \circ (E^a - I) = E^{na} - I = \Delta_{na}$.

LEMMA 4.2 *Given $h \in F(\mathbb{Z})$, then the following identities hold*

1. $(E^{pi} h)_r = E^i h_r$ *for* $i \geq 0$.
2. $(\Delta_p^m h)_r = \Delta^m h_r$ *for* $m \geq 0$.

PROOF. 1. We write $(E^{pi}h)_r(k) = (E^{pi}h)(pk+r) = h(p(k+i)+r) = h_r(k+i) = (E^i h_r)(k)$.
2. We have $\Delta_p^m = (E^p - I)^m = \sum_{i=0}^m (-1)^{m-i}\binom{m}{i} E^{pi}$. Therefore, $(\Delta_p^m h)_r = \sum_{i=0}^m (-1)^{m-i}\binom{m}{i}(E^{pi}h)_r = \sum_{i=0}^m (-1)^{m-i}\binom{m}{i} E^i h_r = (E-I)^m h_r = \Delta^m h_r \square$

PROPOSITION 4.3 *A function $h \in F(\mathbb{Z})$ is quasi-polynomial of period p and degree d if and only if there exists $(c_0, \ldots, c_{p-1}) \neq (0, \ldots, 0)$ such that $(\Delta_p^d h)_r(k) = c_r$ for all $k \gg 0$ and $0 \leq r \leq p-1$.*

PROOF. Follows from lemma 4.2 and [6, lemma 4.1.2] $\square$

COROLLARY 4.4 *For $h \in F(\mathbb{Z})$, if $\prod_{i=1}^d (E^{a_i} - I)(h)(n) = 0$ for $n \gg 0$, then h is quasi-polynomial of period $p \in l\mathbb{N}$ and degree $< d$.*

PROOF. Follows from $\Delta_p^d = (E^p - I)^d = (\prod_{i=1}^d (\sum_{j=0}^{\frac{p}{a_i}-1} E^{ja_i})) \circ (\prod_{i=1}^d (E^{a_i} - I)$ $\square$

EXAMPLE 4.5 Given $m \in \mathbb{Z}$ and $Q(t) \in \mathbb{Q}[t, t^{-1}]$ such that $Q(1) \neq 0$. The function $H_m(Q,.)$ associated with Q is a $P-$quasi-polynomial of period p, where $P = (P_0, .., P_{p-1})$ is given by $P_r = V_{m_r}(U_r, .)$.

REMARK 4.6 Suppose $m > 0$. Then, we have

1. $\deg(H_m(Q,.)) = m - 1$.
2. $m_r > 0 \Rightarrow \deg(P_r) = m_r - 1$ and $c(P_r) = \frac{U_r(1)}{(m_r-1)!}$.
3. If $Q(1) = pQ_r(1) \neq 0$ for all $0 \leq r \leq p-1$, then $H_m(Q,.)$ is uniform of degree $m-1$ and its leading coefficient is $c(H_m(Q,.)) = \frac{Q_r(1)}{(d-1)!} = \frac{Q(1)}{p(d-1)!}$.
4. Suppose $p = qu$ and there exists $K(t) \in \mathbb{Q}[t, t^{-1}]$ such that $K(t^u) = Q(t)$, we set $\xi = e^{\frac{2i\pi}{p}}$. If $Q(\xi^j) = 0$ for $0 < j < q$ and $Q(\xi^q) \neq 0$. Then, the following conditions hold
   (a) $P_r = 0$ if $r \notin u\mathbb{Z}$.
   (b) $\deg(P_r) = m - 1$ and $c(P_r) = \frac{Q_r(1)}{(m-1)!} = \frac{Q(\xi^q)}{p(m-1)!}$ if $r \in u\mathbb{Z}$.

PROPOSITION 4.7 *Let $m > 0$ be an integer and $h \in F(\mathbb{Z})$ be a function satisfying $h(n) = 0$ for $n \ll 0$. We consider $\Phi(t) = \sum_n h(n)t^n$. Then, the following conditions are equivalent*

1. *h is quasi-polynomial of period p and of degree $m-1$.*

2. $(1-t^p)^m\Phi(t) = Q(t) \in \mathbb{Z}[t,t^{-1}]$ *and there exists a p-component Q_r of Q such that $Q_r(1) \neq 0$.*

3. *There exists a unique $Q(t) \in \mathbb{Z}[t,t^{-1}]$ such that $degH_m(Q,.) = m-1$ and $H_m(Q,n) = h(n)$ for $n > \deg(Q) - pm$.*

In particular, h is a uniform quasi-polynomial function of period p and degree $m-1$ if and only if there exists a unique $Q(t) \in \mathbb{Z}[t,t^{-1}]$ such that $Q(1) \neq 0, Q(e^{j\frac{2i\pi}{p}}) = 0$ for all $0 < j < p$ and $H_m(Q,n) = h(n)$ for $n > \deg(Q) - pm$. In this case, the leading coefficient is $c(h) = \frac{Q(1)}{p(m-1)!}$.

PROOF. Assume 1. and set $\Phi_r(t) = \sum_n h_r(n)t^n$ for all $0 \le r \le p-1$. It follows from [6, 4.1.7] that $(1-t)^m\Phi_r(t) = Q_r(t) \in Z[t,t^{-1}]$. Since $\deg(h) = m-1 \ge 0$, there exists $0 \le r \le p-1$ such that $Q_r(1) \neq 0$. Setting $Q(t) = \sum_{r=0}^{p-1} t^r Q_r(t^p)$ we deduce 2.
2. $\Rightarrow$ 3. follows from theorem 2.4.
3. $\Rightarrow$ 1. follows from the definition of H_m.
The particular case follows from lemma 2.6 and remark 4.6 $\square$

COROLLARY 4.8 Let $N(t)$ be an element of $\mathbb{Z}[t,t^{-1}]$ such that $N(1) \neq 0$ and $p \in l\mathbb{N}$. We set $\Phi(t) = \sum_n h(n)t^n$ the expansion of

$$\frac{N(t)}{\prod_{i=1}^d (1-t^{a_i})} = (1-t^p)^{-d}N(t)F_p(t)$$

as a formal Laurent series. Then, $h(n) = H_d(NF_p,n)$ for $n > \deg(N) - \sum_{i=1}^d a_i$. Moreover, if in addition $\gcd(A) = 1$, then $h = H_d(NF_p,.)$ is uniform of degree $d-1$ and its leading coefficient is $c(h) = \frac{N(1)p^{d-1}}{(d-1)!\prod_{i=1}^d a_i}$.

EXAMPLE 4.9 We write $s(n)$ for the number of solutions of the equation $a_1x_1 + .. + a_dx_d = n$ in nonnegative integers we get $s(n) = H_d(F_p,n)$ for all $n \ge 0$ where $p \in l\mathbb{N}$. In particular, if $\gcd(A) = 1$ then $n \mapsto s(n)$ is a uniform quasi-polynomial function of degree $d-1$ and of leading coefficient $c(s) = \frac{p^{d-1}}{(d-1)!\prod_{i=1}^d a_i}$.
For instance, the number of solutions of the equation $2x_1 + 3x_2 = n$ is $s(n) = H_2(F_6,n)$ (see example 2.2).

EXAMPLE 4.10 Let R_0 be a commutative ring and $R = R_0[t_1,\dots,t_d]$. Suppose that R is $\mathbb{Z}$-graded in such a way that every element of R_0 is homogeneous of degree zero and each t_i is homogeneous of degree a_i. Let $M = \oplus_{n\in\mathbb{Z}}M_n$ be a finitely generated graded R-module such that the length $l_{R_0}(M_n)$ of each M_n as an R_0- module is finite. The numerical function $H^0(M,.) : \mathbb{Z} \to \mathbb{Z}, n \mapsto l_{R_0}(M_n)$ is called the Hilbert function of M. The iterated cumulative Hilbert functions are defined by $H^{j+1}(M,n) = \sum_{i=0}^n H^j(M,i)$. The Poincaré series of M is denoted by $P_M(t) = \sum_n H^0(M,n)t^n$. By the Hilbert-Samuel theorem [3, 4.2 Theorem 1] there exists $Q_M(t) \in \mathbb{Z}[t,t^{-1}]$ such that $Q_M(1) \neq 0$ and $P_M(t) = \frac{Q_M(t)}{\prod_{i=1}^d(1-t^{a_i})}$. Moreover, it is known that $H^0(M,.)$ is quasi-polynomial [2]. Given $p \in l\mathbb{N}$ and $j \geq 1$ we set $a_{d+1} = \cdots = a_{d+j} = 1$. So the generating function of the $H^j(M,.)$ is

$$\sum_n H^j(M,n)t^n = \frac{P_M(t)}{(1-t)^j} = \frac{Q_M(t)}{\prod_{i=1}^{d+j}(1-t^{a_i})}.$$

It follows from corollary 4.8 that $H^j(M,n) = H_{d+j}(Q_M F_p, n)$ for all $j \geq 0$ and $n > \deg(Q_M) - \sum a_i$. Moreover, if $j > 0$ or $j = 0$ and $\gcd(A) = 1$ then $H^j(M,.)$ is a uniform quasi-polynomial function of degree $d+j-1$ and of leading coefficient $c(H^j(M,n)) = \frac{Q_M(1)p^{d+j-1}}{(d+j-1)!\prod_{i=1}^{d} a_i}$.

REFERENCES

1. V. Barucci, D. E. Dobbs and M. Fontana, Maximality Properties in Numerical Semigroups and Applications to One-Dimentional Analytically Irreducible Local Domains. Memoirs of the Amer. Math. Soc. vol 125, n○ 598: 1-78, 1997.

2. V.V. Bavula, Identification of the Hilbert function and Poincaré series, and the dimension of modules over filtred rings. Russian Acad. Sci. Izv. Math. 44, n° 2: 225-246, 1995.

3. N. Bourbaki, Algèbre commutative. Chapitre 8, Masson, 1983.

4. A. Brauer, On a problem of partitions, Amer. J. Math. 64: 299-312, 1942.

5. A. Brauer and J. E. Shockley, On a problem of Frobenius. J. Reine Angew. Math. 211: 215-220, 1962.

6. W. Bruns and J. Herzog, Cohen-Macaulay Rings. Cambridge University Press, Cambridge, U.K., 1993.

7. R. Fröberg, The Frobenius number of some semigroups. Comm. in Algebra 22, 14: 6021-6024, 1994.

8. M. Lewin, On a problem of Frobenius for an almost consecutive set of integers. J. Reine Angew. Math. 273: 134-137, 1975.

9. M. Raczunas and P. Chrzastowski-Wachtel, A Diophantine problem of Frobenius in terms of the least common multiple. Discrete Mathematics, 150: 347-357, 1996.

10. O.J. Rödseth, On a linear Diophantine problem of Frobenius. J. Reine Angew. Math. 301: 171-178, 1978.

11. E.S. Selmer, A linear Diophantine problem of Frobenius. J. Reine Angew. Math. 293/294: 1-17, 1977.